BETHE ALGEBRA OF GAUDIN MODEL, CALOGERO-MOSER SPACE AND CHEREDNIK ALGEBRA

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Abstract. We identify the Bethe algebra of the Gaudin model associated to $\mathfrak{gl}_N$ acting on a suitable representation with the center of the rational Cherednik algebra and with the algebra of regular functions on the Calogero-Moser space.

1. Introduction

The Bethe algebra of the Gaudin model associated to $\mathfrak{gl}_N$ is a remarkable commutative subalgebra of the universal enveloping algebra of the current algebra of $\mathfrak{gl}_N$. It is also known by the names of the algebra of higher Gaudin Hamiltonians, see [FFR], or the algebra of higher transfer matrices, see [MTV1], or the quantum shift of argument subalgebra, see [FFRb].

The Bethe algebra acts on a subspace M of a given $\mathfrak{gl}_N$-weight of any $\mathfrak{gl}_N[t]$-module producing a commutative family of linear operators $\mathcal{B}(M) \in \operatorname{End} M$. The main problem of the Gaudin model is to describe common eigenvectors and eigenvalues of this family.

It often turns out that the Bethe algebra $\mathcal{B}(M)$ can be naturally identified with the algebra of regular functions $\mathcal{O}_{\mathcal{X}}$ on an affine variety $\mathcal{X}$, and M becomes the regular representation of $\mathcal{O}_{\mathcal{X}}$. Then the common eigenvectors of $\mathcal{B}(M)$ are in a bijective correspondence with the points of $\mathcal{X}$, the joint spectrum of $\mathcal{B}(M)$ is simple and $\mathcal{B}(M)$ is a maximal commutative subalgebra of $\operatorname{End} M$. One can also hope to get new information about the variety $\mathcal{X}$ by studying the algebra $\mathcal{B}(M)$.

Such an idea was realized in [MTV2], [MTV3], where M is a subspace of a given weight of an arbitrary finite-dimensional irreducible $\mathfrak{gl}_N[t]$-module. Then $\mathcal{B}(M)$ is a finite-dimensional algebra and $\mathcal{X}$ is the scheme-theoretic intersection of a suitable Schubert varieties in a Grassmannian of N planes. Besides the new information on the spectrum of the Gaudin model, on the algebro-geometric side this study gave a proof of the B. and M. Shapiro conjecture, see [MTV5], a proof of the transversality conjecture discussed in [S], and an effective proof of the reality of the Schubert calculus.

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In this paper, we consider the action of $\mathfrak{gl}_N[t]$ on polynomials in several variables with values in a tensor product of vector representations of $\mathfrak{gl}_N$, which is defined via the evaluation map, and the subspace M of polynomials with values in the zero sl_N-weight subspace of the tensor product. Thus, in contrast to [MTV2], [MTV3], we keep the evaluation parameters and the parameters in the Bethe algebra formal variables, which makes M an infinite-dimensional module. In this case, we show that the corresponding affine variety $\mathcal{X}$ is the Calogero-Moser space. The Calogero-Moser is a celebrated affine non-singular variety, which appears in many areas of mathematics, see [KKS], [Wi], [EG], [CBH].

We prove two main theorems. First, we show that $\mathcal{B}(M)$ is naturally isomorphic to the center of the rational Cherednik algebra of type A, see Theorem 2.9. Second, we show that $\mathcal{B}(M)$ is naturally isomorphic to the algebra $\mathcal{O}_{\mathcal{X}}$ of regular functions on the Calogero-Moser space, see Theorem 4.3.

The Bethe algebra $\mathcal{B}(M)$ is generated by coefficients of a row determinant of some matrix, see [CT], [MTV1]. We show that the center of the Cherednik algebra is generated by coefficients of an explicit determinant-like formula and that the algebra $\mathcal{O}_{\mathcal{X}}$ of regular functions on the Calogero-Moser space is generated by the coefficients of the polynomial version of the Wilson Ψ function. The isomorphisms in Theorems 2.9 and 4.3 just send the corresponding coefficients to each other. The proof of Theorem 2.9 is a simple algebraic argument. The proof of Theorem 4.3 follows the logic of [MTV2] and it is more involved. In particular, we use the machinery of the Bethe ansatz and the Wilson correspondence of the Calogero-Moser space to the adelic Grassmannian.

It is proved in [EG] that the center of the Cherednik algebra of type A is isomorphic to the algebra of regular functions on the Calogero-Moser space. We recover this result.

The paper is organized as follows. We start with identifying the Bethe algebra with the center of the Cherednik algebra in Section 2. We discuss this result in Section 3. In particular, we give an explanation of Theorem 2.9 using a general construction of a commutative subalgebra from the center of an algebra, see Sections 3.1 and 3.2. In Section 4 we describe the map between the Bethe algebra and the algebra of regular functions on the Calogero-Moser space. We give the proof that this map is a well-defined isomorphism of algebras in Section 5. Some corollaries of this isomorphism are given in Section 6. In particular, Section 6.2 describes the bijections between eigenvectors of the Bethe algebra $\mathcal{B}(M)$ and three sets which are known to be equivalent: points of the Calogero-Moser space, the points of the adelic Grassmannian and the set of irreducible representations of the Cherednik algebra. These bijections follow from Theorems 2.9 and 4.3.

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2. Bethe algebra of the Gaudin model and the center of the Cherednik algebra

2.1. **Multi-symmetric polynomials.** Let $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}] = \mathbb{C}[z_1,\dots,z_N,\lambda_1,\dots,\lambda_N]$ be the algebra of polynomials in commuting variables.

Let S_N be the group of permutations of N elements. We often consider actions of the group S_N which permute indices in the groups of N variables, in such cases we will indicate the affected group of variables by the upper indices. For example, S_N^z permutes the variables $z_1,\dots,z_N$ and no other variables, $S_N^{z,\lambda}$ permutes the variables $z_1,\dots,z_N$ and the variables $\lambda_1,\dots,\lambda_N$, etc. We use the same notation for the elements of S_N. For example, for $\sigma,\tau\in S_N$,

$$(\sigma^z\tau^\lambda)\big(p(z_1,\dots,z_N,\lambda_1,\dots,\lambda_N)\big) = p\big(z_{\sigma(1)},\dots,z_{\sigma(N)},\lambda_{\tau(1)},\dots,\lambda_{\tau(N)}\big).$$

We also have $\sigma^z\tau^\lambda = \tau^\lambda\sigma^z$ and $\sigma^z\sigma^\lambda = \sigma^{z,\lambda}$.

Let $P_N = \mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]^{S_N^{z,\lambda}} \subset \mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]$ be the algebra of polynomials invariant with respect to simultaneous permutations of $z_1,\dots,z_N$ and $\lambda_1,\dots,\lambda_N$. We call P_N the *algebra of multi-symmetric polynomials*. The algebra P_N is also known by the names of MacMahon polynomials, vector symmetric polynomials, diagonally symmetric polynomials, etc, it is well-studied, see for example [W].

Consider the algebras $\mathbb{C}[\boldsymbol{z}]^{S_N^z}$ and $\mathbb{C}[\boldsymbol{\lambda}]^{S_N^\lambda}$ of symmetric polynomials in $\boldsymbol{z}$ and $\boldsymbol{\lambda}$ respectively. We have an obvious inclusion $\mathbb{C}[\boldsymbol{z}]^{S_N^z}\otimes\mathbb{C}[\boldsymbol{\lambda}]^{S_N^\lambda} \to P_N$ given by the multiplication map. The following lemma is a standard fact.

Lemma 2.1. *The algebra P_N is a free $\mathbb{C}[\boldsymbol{z}]^{S_N^z}\otimes\mathbb{C}[\boldsymbol{\lambda}]^{S_N^\lambda}$-module of rank $N!$.* □

Consider the wreath product $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]\ltimes\mathbb{C}S_N^{z,\lambda}$. We write the elements of $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]\ltimes\mathbb{C}S_N^{z,\lambda}$ in the form $\sum_{\sigma\in S_N} p_\sigma(\boldsymbol{z},\boldsymbol{\lambda})\sigma$, where $p_\sigma(\boldsymbol{z},\boldsymbol{\lambda})\in\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]$. Such an element is zero if and only if all $p_\sigma=0$. The algebra $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]$ is embedded in $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]\ltimes\mathbb{C}S_N^{z,\lambda}$ by the map $p(\boldsymbol{z},\boldsymbol{\lambda})\mapsto p(\boldsymbol{z},\boldsymbol{\lambda})\ id$.

The following is another standard fact.

Lemma 2.2. *The algebra $P_N\cdot id$ is the center of $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]\ltimes\mathbb{C}S_N^{z,\lambda}$.* □

Define the *universal multi-symmetric polynomial*

$$\mathcal{P}^P = \prod_{i=1}^N\big((u-z_i)(v-\lambda_i)-1\big).$$

It is a polynomial in variables u,v with coefficients in P_N. Write

$$\mathcal{P}^P = \sum_{i,j=0}^N p_{ij}u^{N-i}v^{N-j}, \qquad p_{ij}=p_{ij}(\boldsymbol{z},\boldsymbol{\lambda})\in P_N.$$

We have $p_{00}=1$.

Lemma 2.3. *The polynomials p_{ij}, $i,j=0,1,\dots,N$, generate the algebra P_N.*

Proof. Let $Z=\operatorname{diag}(z_1,\dots,z_N)$, $\Lambda=\operatorname{diag}(\lambda_1,\dots,\lambda_N)$ be the diagonal $N\times N$ matrices. Then $\mathcal{P}^P=\det((u-Z)(v-\Lambda)-1)$ and

$$\det(u-Z)=u^N+\sum_{i=1}^N p_{i0}u^{N-i},\qquad \det(v-\Lambda)=v^N+\sum_{j=1}^N p_{0j}v^{N-j}.$$

Therefore, the coefficients of the series $\log\det(1-(u-Z)^{-1}(v-\Lambda)^{-1})$ in u^{-1},v^{-1} are polynomials in p_{ij}. We have

$$\log\det(1-(u-Z)^{-1}(v-\Lambda)^{-1}) \;=\; \operatorname{tr}(\log(1-(u-Z)^{-1}(v-\Lambda)^{-1})) \;=$$
$$-\sum_{r=1}^\infty \frac{1}{r}\operatorname{tr}\left((\sum_{i=0}^\infty Z^i u^{-i-1})(\sum_{j=0}^\infty \Lambda^j v^{-j-1})\right)^r \;=\; -\sum_{i,j,k,l=0}^\infty c_{ij}^{kl}\;\operatorname{tr}(\Lambda^k Z^l)u^{-i-1}v^{-j-1}\,,$$

where c_{ij}^{kl} are rational numbers. In the last equality we used $Z\Lambda=\Lambda Z$. Then $c_{ij}^{ij}=1$ and $c_{ij}^{kl}=0$ if $k>i$ or $l>j$. Therefore, by triangularity, $\operatorname{tr}(\Lambda^k Z^l)$ are polynomials in p_{ij}.

It is well-known that the power sums multi-symmetric polynomials $\operatorname{tr}(\Lambda^k Z^l)=\sum_{i=1}^N\lambda_i^k z_i^l$ generate P_N, see [W]. The lemma follows. □

2.2. **The Bethe algebra.** Let $\mathfrak{gl}_N$ denote the complex Lie algebra of all $N\times N$ matrices and $U\mathfrak{gl}_N$ its universal enveloping algebra. The algebra $U\mathfrak{gl}_N$ is generated by the elements e_{ij}, $i,j=1,\dots,N$, satisfying the relations $[e_{ij},e_{sk}]=\delta_{js}e_{ik}-\delta_{ik}e_{sj}$.

Let $\mathfrak{gl}_N[t]$ denote the current algebra of $\mathfrak{gl}_N$ and $U(\mathfrak{gl}_N[t])$ its universal enveloping algebra. The algebra $U(\mathfrak{gl}_N[t])$ is generated by elements $e_{ij}\otimes t^r$, $i,j=1,\dots,N$, $r\in\mathbb{Z}_{\geqslant 0}$, satisfying the relations $[e_{ij}\otimes t^r, e_{sk}\otimes t^p]=\delta_{js}e_{ik}\otimes t^{r+p}-\delta_{ik}e_{sj}\otimes t^{r+p}$.

It is convenient to collect elements of $\mathfrak{gl}_N[t]$ in generating series of the variable u. Namely, for $g\in\mathfrak{gl}_N$, set

$$g(u)=\sum_{s=0}^\infty (g\otimes t^s)u^{-s-1}.$$

Let $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_N)$ be a sequence of formal commuting variables. Denote the algebra of polynomials in variables $\lambda_1,\dots,\lambda_N$ with values in $U(\mathfrak{gl}_N[t])$ by $U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}]$.

We define the *row determinant* of an $N\times N$ matrix A with entries a_{ij} in a possibly non-commutative algebra to be

$$\operatorname{rdet}A \;=\; \sum_{\sigma\in S_N} a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{N\sigma(N)}. \tag{2.1}$$

Denote the operator of the formal differentiation with respect to the variable u by ∂. Define the *universal operator* $\mathcal{D}^{\mathcal{B}}$ by

$$\mathcal{D}^{\mathcal{B}} = \operatorname{rdet}\begin{pmatrix} \partial-\lambda_1-e_{11}(u) & -e_{21}(u) & \dots & -e_{N1}(u) \\ -e_{12}(u) & \partial-\lambda_2-e_{22}(u) & \dots & -e_{N2}(u) \\ \dots & \dots & \dots & \dots \\ -e_{1N}(u) & -e_{2N}(u) & \dots & \partial-\lambda_N-e_{NN}(u) \end{pmatrix}.$$

The universal operator $\mathcal{D}^{\mathcal{B}}$ is a differential operator in the variable u whose coefficients are formal power series in u^{-1} with coefficients in $U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}]$. Write

$$\mathcal{D}^{\mathcal{B}} = \partial^N + \sum_{i=1}^N B_i(u)\partial^{N-i}, \qquad B_i(u) = \sum_{j=0}^\infty B_{ij}u^{-j}, \qquad B_{ij} \in U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}].$$

We call the unital subalgebra of $U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}]$ generated by B_{ij}, $i=1,\dots,N$, $j \in \mathbb{Z}_{\geqslant 0}$, the *Bethe algebra* and denote it by $\mathcal{B}_N$.

Lemma 2.4 ([CT],[MTV1]). *The algebra $\mathcal{B}_N$ is commutative. The algebra $\mathcal{B}_N$ commutes with e_{ii} and multiplications by λ_i, $i=1,\dots,N$.* $\square$

As a subalgebra of $U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}]$, $\mathcal{B}_N$ acts on any $U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}]$-module M. Since $\mathcal{B}_N$ commutes with e_{ii}, it preserves the $\mathfrak{gl}_N$-weight decomposition of the module M.

2.3. **The Cherednik algebra.** Denote by H_N the *rational Cherednik algebra associated with the symmetric group S_N*. The algebra H_N is the unital complex algebra with generators x_i, y_i, s_{jk}, where $i,j,k=1,\dots,N$, $j\neq k$, and relations

$$\begin{gathered} s_{ij}=s_{ji}, \quad s_{ij}^2=1, \quad s_{ij}s_{jk}=s_{ik}s_{ij}, \quad s_{ij}s_{kl}=s_{kl}s_{ij}, \\ [x_i,x_j]=[y_i,y_j]=0, \\ s_{ij}x_i=x_js_{ij}, \quad s_{ij}y_i=y_js_{ij}, \quad [s_{ij},x_k]=[s_{ij},y_k]=0, \\ [x_i,y_j]=s_{ij}, \quad [x_i,y_i]=-\sum_{a,\ a\neq i} s_{ia}, \end{gathered}$$

where in each relation all the indices are distinct elements of $\{1,\dots,N\}$. The rational Cherednik algebra H_N is a two step degeneration of the double affine Hecke algebra, see [C].

We employ the notation $\mathbb{C}[\boldsymbol{x}]=\mathbb{C}[x_1,\dots,x_N]$ and $\mathbb{C}[\boldsymbol{y}]=\mathbb{C}[y_1,\dots,y_N]$. The algebra H_N is a deformation of the wreath product of the algebra of polynomials in commuting variables $\mathbb{C}[\boldsymbol{x}]\otimes\mathbb{C}[\boldsymbol{y}]$ and of the group algebra $\mathbb{C}[S_N^{x,y}]$ generated by transpositions s_{ij}. In particular, we have a linear isomorphism given by the multiplication map:

$$\begin{gathered} \mathbb{C}[\boldsymbol{x}]\otimes\mathbb{C}[S_N]\otimes\mathbb{C}[\boldsymbol{y}] \;\to\; H_N, \\ q\otimes\sigma\otimes p \;\mapsto\; q\;\sigma\;p. \end{gathered}$$

We call this isomorphism the *normal ordering map* and denote it by ::.

For example, $: y_1x_2y_3x_4 := x_2x_4y_1y_3$, $: x_1y_1^2x_2 := x_1x_2y_1^2$, etc. Note that we omit the tensor signs in writing the elements of $\mathbb{C}[\boldsymbol{x}]\otimes\mathbb{C}[S_N]\otimes\mathbb{C}[\boldsymbol{y}]$.

Denote by $\mathcal{Z}_N\subset H_N$ the center of H_N.

Define the *universal central polynomial*

$$\mathcal{P}^{\mathcal{Z}}=(-1)^N\sum_{\sigma\in S_N}:\prod_{i,\ \sigma(i)=i}(1-(v-x_i)(u-y_i)):\ (-1)^\sigma\sigma.$$

The polynomial $\mathcal{P}^{\mathcal{Z}}$ is a polynomial in u and v with coefficients in H_N. Write

$$\mathcal{P}^{\mathcal{Z}}=\sum_{i,j=0}^{N}c_{ij}\ v^{N-i}u^{N-j},\qquad c_{ij}\in H_N.$$

Theorem 2.5. *The elements c_{ij}, $i,j=0,\dots,N$, generate the center $\mathcal{Z}_N\subset H_N$.*

Proof. First, we show that the elements c_{ij} are central. It is clear that $s_{ij}\mathcal{P}^{\mathcal{Z}}=\mathcal{P}^{\mathcal{Z}}s_{ij}$ for any i,j. Hence, to show that $[x_i,\mathcal{P}^{\mathcal{Z}}]=[y_i,\mathcal{P}^{\mathcal{Z}}]=0$ for all $i=1,\dots,N$, it is enough to check these equalities for $i=1$ only.

We begin with proving that $[y_1,\mathcal{P}^{\mathcal{Z}}]=0$. Let

$$a_i=(1-(v-x_i)(u-y_i)),\qquad A_\sigma=:\prod_{i,\ \sigma(i)=i}a_i:,$$

so that

$$\mathcal{P}^{\mathcal{Z}}=(-1)^N\sum_{\sigma\in S_N}(-1)^\sigma A_\sigma\sigma\,.$$

Let

$$[y_1,\mathcal{P}^{\mathcal{Z}}]=(-1)^N\sum_{\sigma\in S_N}(-1)^\sigma C_\sigma\sigma,$$

where C_σ has the form

$$C_\sigma=\sum_{i=1}^{k_\sigma}p_{i,\sigma}(\boldsymbol{x})q_{i,\sigma}(\boldsymbol{y}).$$

Then we have

$$C_\sigma=A_\sigma(y_1-y_{\sigma(1)})-\sum_{i=2}^{N}\phi([y_1,A_{s_{1i}\sigma}]s_{1i}),$$

where ϕ is a linear map

$$\phi: H_N\to H_N,\qquad p(\boldsymbol{x})q(\boldsymbol{y})\sigma\mapsto\delta_{id,\sigma}p(\boldsymbol{x})q(\boldsymbol{y}).$$

The expression $\phi([y_1,A_{s_{1i}\sigma}]s_{1i})$ equals zero unless $\sigma(1)=i$ or $\sigma(i)=1$. Hence, $C_\sigma=0$ if $\sigma(1)=1$.

Assume that $\sigma(1) = k \neq 1$ and $\sigma(k) = 1$. Then

$$\begin{aligned} C_\sigma \,&=\, A_\sigma(y_1 - y_k) - \phi([y_1, A_{s_{1k}\sigma}]s_{1k}) \,= \\ &=\, A_\sigma(y_1 - y_k) - (A_\sigma(u - y_k) - A_\sigma(u - y_1) - \\ &\qquad - (v - x_1)A_\sigma(u - y_1)(u - y_k) + (v - x_1)A_\sigma(u - y_1)(u - y_k)) \,=\, 0. \end{aligned}$$

Assume that $\sigma(1) = k \neq 1$, $\sigma(l) = 1$, and $k \neq l$. Then

$$\begin{aligned} C_\sigma \,&=\, A_\sigma(y_1 - y_k) - \phi([y_1, A_{s_{1k}\sigma}]s_{1k}) - \phi([y_1, A_{s_{1l}\sigma}]s_{1l}) \,= \\ &=\, A_\sigma(y_1 - y_k) - A_\sigma(u - y_k) + A_\sigma(u - y_1) \,=\, 0. \end{aligned}$$

The proof of $[x_1, \mathcal{P}^{\mathcal{Z}}] = 0$ is similar with the following modification: we use

$$\mathcal{P}^{\mathcal{Z}} = (-1)^N \sum_{\sigma\in S_N} (-1)^\sigma \sigma A_\sigma$$

and move elements $\sigma \in S_N$ to the left.

There is a filtration on H_N given by $\deg x_i = \deg y_i = 1$, $\deg s_{ij} = 0$. The associated graded ring is isomorphic to the wreath product $\mathbb{C}[\boldsymbol{z}, \boldsymbol{\lambda}] \ltimes \mathbb{C}S_N^{z,\lambda}$, and all graded components are finite-dimensional. The center of H_N is projected into the center of the wreath product, which is the algebra P_N, see Lemma 2.2. The elements c_{ij} are projected to the elements p_{ij}, which are generators of P_N by Lemma 2.3. Hence, the elements c_{ij} generate the center of H_N. $\square$

A *central character* of H_N is an algebra homomorphism $\chi : \mathcal{Z}_N \to \mathbb{C}$. Central characters determine irreducible representations of H_N, see Theorem 1.24 in [EG]. Let

$$e = \frac{1}{N!} \sum_{\sigma\in S_N} \sigma \in H_N \tag{2.2}$$

be the symmetrizing element.

Theorem 2.6 ([EG])**.** *Any irreducible H_N-module has dimension $N!$ and is isomorphic to the regular representation of S_N as an S_N-module. Irreducible H_N-modules are in a bijective correspondence with algebra homomorphisms $\chi : \mathcal{Z}_N \to \mathbb{C}$. The irreducible H_N-module corresponding to the central character χ is given by $H_N e \otimes_{\mathcal{Z}_N} \chi$.* $\square$

We denote the set of isomorphism classes of irreducible modules of the Cherednik algebra H_N by R_N.

2.4. **The space $\mathcal{V}_1$.** Let V be the vector representation of $\mathfrak{gl}_N$, $\dim V = N$. Let $\epsilon_1, \dots, \epsilon_N$ be the standard basis of V, $e_{ij}\epsilon_k = \delta_{jk}\epsilon_i$.

Let $\mathcal{V}$ be the space of polynomials in commuting variables $z_1, \dots, z_N$ and $\lambda_1, \dots, \lambda_N$ with coefficients in $V^{\otimes N}$:

$$\mathcal{V} = V^{\otimes N} \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{z}, \boldsymbol{\lambda}].$$

For $v \in V^{\otimes N}$ and $p(\boldsymbol{z}, \boldsymbol{\lambda}) \in \mathbb{C}[\boldsymbol{z}, \boldsymbol{\lambda}]$, we write $p(\boldsymbol{z}, \boldsymbol{\lambda})\, v$ instead of $v \otimes p(\boldsymbol{z}, \boldsymbol{\lambda})$.

Denote the subspace of $V^{\otimes N}$ of $\mathfrak{gl}_N$-weight $(1, 1, \dots, 1)$ by $(V^{\otimes N})_{\boldsymbol{1}}$:

$$(V^{\otimes N})_{\boldsymbol{1}} = \{ v \in V^{\otimes N} \mid e_{ii} v = v,\ i = 1, \dots, N \}.$$

We have $\dim (V^{\otimes N})_{\boldsymbol{1}} = N!$. A basis of $(V^{\otimes N})_{\boldsymbol{1}}$ is given by vectors

$$\epsilon_\sigma = \epsilon_{\sigma(1)} \otimes \dots \otimes \epsilon_{\sigma(N)}\,, \qquad \sigma \in S_N\,.$$

Let $\mathcal{V}_{\boldsymbol{1}} = (V^{\otimes N})_{\boldsymbol{1}} \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{z}, \boldsymbol{\lambda}]$.

Consider the space $\mathcal{V}$ as a $U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}]$-module with the series $g(u),\ g \in \mathfrak{gl}_N$, acting by

$$g(u)\, \bigl(p(\boldsymbol{z}, \boldsymbol{\lambda})\, v_1 \otimes \dots \otimes v_N \bigr) \,=\, p(\boldsymbol{z}, \boldsymbol{\lambda}) \sum_{i=1}^{N} \frac{v_1 \otimes \dots \otimes g v_i \otimes \dots \otimes v_N}{u - z_i}\,.$$

and the algebra $\mathbb{C}[\boldsymbol{\lambda}]$ acting by multiplication operators. In particular, the Bethe algebra $\mathcal{B}_N$ acts on $\mathcal{V}$. Since $\mathcal{B}_N$ commutes with e_{ii}, the Bethe algebra $\mathcal{B}_N$ also acts on the space $\mathcal{V}_{\boldsymbol{1}}$. We denote the image of $\mathcal{B}_N$ in $\mathrm{End}(\mathcal{V}_{\boldsymbol{1}})$ by $\bar{\mathcal{B}}_N$.

We compute the action of the coefficients of the universal operator $\mathcal{D}^{\mathcal{B}}$ in $\mathcal{V}_{\boldsymbol{1}}$. Let $\bar{B}_{ij} \in \mathrm{End}(\mathcal{V}_{\boldsymbol{1}})$ be the images of the operators B_{ij}.

Define the *universal Bethe polynomial* $\mathcal{P}^{\bar{\mathcal{B}}}$ by the formula

$$\mathcal{P}^{\bar{\mathcal{B}}} = w(u, \boldsymbol{z}) \left(v^N + \sum_{i=1}^{N} \sum_{j=0}^{\infty} \bar{B}_{ij} u^{-j} v^{N-i} \right), \qquad w(u, \boldsymbol{z}) = \prod_{i=1}^{N} (u - z_i).$$

The universal Bethe polynomial is a polynomial in u and v with coefficients in $\mathrm{End}(\mathcal{V}_{\boldsymbol{1}})$, see [MTV4]. Write

$$\mathcal{P}^{\bar{\mathcal{B}}} = \sum_{i,j=0}^{N} \bar{b}_{ij} u^{N-i} v^{N-j}, \qquad \bar{b}_{ij} \in \mathrm{End}(\mathcal{V}_{\boldsymbol{1}}).$$

Lemma 2.7. *The algebra $\bar{\mathcal{B}}_N$ is generated by $\bar{b}_{ij},\ i, j = 0, 1, \dots, N$.*

Proof. We have

$$w(u, \boldsymbol{z}) = \sum_{i=0}^{N} \bar{b}_{i0} u^{N-i}.$$

Therefore the coefficients of the power series $\mathcal{P}^{\bar{\mathcal{B}}} / w(u, \boldsymbol{z})$ are in the algebra generated by $\bar{b}_{ij}$. □

Lemma 2.8. *For all $\tau \in S_N$, we have:*

$$\mathcal{P}^{\bar{\mathcal{B}}} \epsilon_\tau = (-1)^N \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i,\ \sigma(i) = i} \bigl(1 - (u - z_{\tau^{-1}(i)})(v - \lambda_i)\bigr)\ \epsilon_{\sigma\tau}.$$

Moreover, the operators $\bar b_{ij}$ commute with multiplications by λ_s and z_s.

Proof. Following [MTV4], to compute the polynomial $(-1)^N\mathcal{P}^{\bar B}$ we have to consider the determinant

$$\mathrm{rdet}\begin{pmatrix} \sum_{i=1}^N \frac{e_{11}^{(i)}}{u-z_i}-(v-\lambda_1) & \sum_{i=1}^N \frac{e_{21}^{(i)}}{u-z_i} & \dots & \sum_{i=1}^N \frac{e_{N1}^{(i)}}{u-z_i} \\ \sum_{i=1}^N \frac{e_{12}^{(i)}}{u-z_i} & \sum_{i=1}^N \frac{e_{22}^{(i)}}{u-z_i}-(v-\lambda_2) & \dots & \sum_{i=1}^N \frac{e_{N2}^{(i)}}{u-z_i} \\ \dots & \dots & \dots & \dots \\ \sum_{i=1}^N \frac{e_{1N}^{(i)}}{u-z_i} & \sum_{i=1}^N \frac{e_{2N}^{(i)}}{u-z_i} & \dots & \sum_{i=1}^N \frac{e_{NN}^{(i)}}{u-z_i}-(v-\lambda_N) \end{pmatrix},$$

expand it, ignore all the terms with poles of order higher than one, compute the action of $e_{jk}^{(i)}$, and finally multiply by $w(u,\boldsymbol{z})$. Here $e_{jk}^{(i)}$ denote the operators e_{jk} acting on the i-th factor in $V^{\otimes N}$.

The terms A_σ in the expansion are labeled by permutations $\sigma\in S_N$, see (2.1), and

$A_\sigma=(e_{\sigma(1)1}(u)-\delta_{\sigma(1)1}(v-\lambda_1))(e_{\sigma(2)2}(u)-\delta_{\sigma(2)2}(v-\lambda_2))\dots(e_{\sigma(N)N}(u)-\delta_{\sigma(N)N}(v-\lambda_N)).$

Note also that

$$e_{ij}(u)\epsilon_\tau=\frac{e_{ij}^{(\tau^{-1}(j))}}{u-z_{\tau^{-1}(j)}}\ \epsilon_\tau.$$

Therefore

$$A_\sigma\epsilon_\tau=\prod_{i,\ \sigma(i)=i}\left(\frac{1}{u-z_{\tau^{-1}(i)}}-(v-\lambda_i)\right)\ \epsilon_{\sigma\tau}+\dots,$$

where the dots denote the terms with at least one pole in u of order greater than 1.

The lemma follows. $\square$

2.5. **The Bethe algebra and the center of the Cherednik algebra.** We identify the space $\mathcal{V}_{\mathbf 1}$ with H_N as follows. Let ι be the isomorphism of vector spaces given by

$$\iota:\ \mathcal{V}_{\mathbf 1}\to H_N,\qquad p(\boldsymbol{z})q(\boldsymbol{\lambda})\epsilon_\tau\mapsto q(\boldsymbol{x})\tau p(\boldsymbol{y}),$$

for all $\tau\in S_N$ and all polynomials p,q.

The map ι identifies the action of the Bethe algebra $\bar{\mathcal{B}}_N$ in $\mathcal{V}_{\mathbf 1}$ with the action of the center $\mathcal{Z}_N$ in H_N. Namely, we have the following theorem.

Theorem 2.9. *We have*

$$\iota\ \mathcal{P}^{\bar{\mathcal{B}}}=\mathcal{P}^{\mathcal{Z}}\iota.$$

Proof. It is sufficient to check the equality in the theorem on elements of $\mathcal{V}_{\mathbf 1}$ of the form $p(\boldsymbol{z})q(\boldsymbol{\lambda})\epsilon_\tau$, where $\tau\in S_N$ and p,q are polynomials.

The left hand side, $\iota\ \mathcal{P}^{\bar{\mathcal{B}}}(p(\boldsymbol{z})q(\boldsymbol{\lambda})\epsilon_\tau)$, is computed using Lemma 2.8.

For the right hand side, we have

$$\mathcal{P}^{\mathcal{Z}}\iota(p(\boldsymbol{z})q(\boldsymbol{\lambda})\epsilon_\tau)=\mathcal{P}^{\mathcal{Z}}q(\boldsymbol{x})\tau p(\boldsymbol{y})=q(\boldsymbol{x})\mathcal{P}^{\mathcal{Z}}\tau p(\boldsymbol{y}),$$

where in the second equality we used that $\mathcal{P}^{\mathcal{Z}}$ is central by Theorem 2.5.

Furthermore, if $\sigma(i) = i$ then

$$q(\boldsymbol{x})(v - x_i)(u - y_i)\sigma\tau p(\boldsymbol{y}) = q(\boldsymbol{x})(v - \lambda_i)\sigma\tau(u - y_{\tau^{-1}(i)})p(\boldsymbol{y}).$$

The theorem follows. $\square$

Corollary 2.10. *There exists a unique isomorphism of algebras $\tau_{BZ} : \bar{\mathcal{B}}_N \to \mathcal{Z}_N$ which maps $\bar{b}_{ij}$ to c_{ij}.* $\square$

3. Remarks on Theorem 2.9

3.1. **A construction of a commutative algebra.** We describe a useful way to construct commutative subalgebras from the center of an algebra.

Let A be an algebra and let $\mathcal{Z}_A \subset A$ be the center of A. Assume that $A_+, A_- \subset A$ are subalgebras of A such that $A = A_+A_-$. By that we mean that the multiplication map $A_+ \otimes A_- \to A$ is an isomorphism of vector spaces.

Let A_+^{op} be the algebra A_+ with the opposite multiplication: $A_+^{op} = A_+$ as vector spaces and the multiplication map $A_+^{op} \otimes A_+^{op} \to A_+^{op}$ sends $a_+ \otimes b_+$ to b_+a_+.

We have a unique isomorphism of vector spaces defined by

$$\alpha : A \to A_+^{op} \otimes A_-, \qquad a_+a_- \mapsto a_+ \otimes a_-,$$

for all $a_+ \in A_+, a_- \in A_-$.

Lemma 3.1. *The algebra $\alpha(\mathcal{Z}_A)$ is a commutative subalgebra of $A_+^{op} \otimes A_-$ isomorphic to $\mathcal{Z}_A$.*

Proof. Let $a = \sum_i a_+^{(i)} a_-^{(i)}$ and $b = \sum_j b_+^{(j)} b_-^{(j)}$ be elements of the center $\mathcal{Z}_A$. Here $a_\pm^{(i)}, b_\pm^{(j)} \in A_\pm$. We have

$$\alpha^{-1}([\alpha(a), \alpha(b)]) = \sum_{ij}([b_+^{(j)}, a_+^{(i)}]a_-^{(i)}b_-^{(j)} + a_+^{(i)}, b_+^{(j)}[a_-^{(i)}b_-^{(j)}]) =$$
$$\sum_{ij}(-a_+^{(i)}[b_+^{(j)}, a_-^{(i)}]b_-^{(j)} + a_+^{(i)}, b_+^{(j)}[a_-^{(i)}b_-^{(j)}]) = 0.$$

Here the first equality follows from the definitions, the second from the centrality of a and the third one from the centrality of b. $\square$

Remark 3.2. The idea of this construction can be eventually traced back to Kostant-Adler method in the theory of integrable systems, see [K], [ReS]. A similar idea in a disguised form is involved in the factorization method, see [RS] and the construction of higher Gaudin Hamiltonians, see [FFR], [FFRb].

3.2. **Another form of Theorem 2.9.** We interpret Theorem 2.9 as a coincidence of two natural commutative subalgebras in the algebra $\mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}[S_N^y])$.

Let $A = H_N$, $A_+ = \mathbb{C}[\boldsymbol{x}] = A_+^{op}$ and $A_- = \mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}[S_N^y]$. Then by Lemma 3.1 we have a commutative subalgebra $\alpha(\mathcal{Z}_N) \subset \mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}[S_N^y])$.

Let

$$U_{\mathbf{0}} = \{g \in U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}] \mid \ ge_{ii} = e_{ii}g, \ i = 1, \dots, N\}$$

be the subalgebra of $U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}]$ of $\mathfrak{gl}_N$-weight $\mathbf{0} = (0, \dots, 0)$. Note that $\mathcal{B}_N \subset U_{\mathbf{0}}$.

If M is a $\mathfrak{gl}_N[t]$-module, then $U_{\mathbf{0}}$ preserves the $\mathfrak{gl}_N$-weight decomposition of M. In particular, $U_{\mathbf{0}}$ acts on $\mathcal{V}_{\mathbf{1}}$.

Identify the space $\mathcal{V}_{\mathbf{1}}$ with the space $\mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}S_N^y)$ by the linear isomorphism $\tilde{\iota}$:

$$\tilde{\iota}: \ \mathcal{V}_{\mathbf{1}} \to \mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}S_N^y), \qquad p(\boldsymbol{z})q(\boldsymbol{\lambda})\epsilon_\tau \mapsto q(\boldsymbol{x}) \otimes (\tau p(\boldsymbol{y})),$$

for all $\tau \in S_N$ and all polynomials p, q.

Lemma 3.3. *The map $\tilde{\iota}$ identifies the image of the algebra $U_{\mathbf{0}}$ in $\mathrm{End}(\mathcal{V}_{\mathbf{1}})$ with the algebra $\mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}S_N^y)$ acting by left multiplications.*

Proof. The right multiplication by x_i and y_j correspond to multiplications by λ_i and z_j in the space $\mathcal{V}_{\mathbf{1}}$. The right multiplication by s_{ij} corresponds to switching the i-th with the j-th factors and z_i with z_j in the space $\mathcal{V}_{\mathbf{1}}$. Clearly, the algebra $U_{\mathbf{0}}$ commutes with all these operators. Therefore, the map $\tilde{\iota}$ identifies the image of the algebra $U_{\mathbf{0}}$ in $\mathrm{End}(\mathcal{V}_{\mathbf{1}})$ with a subalgebra of left multiplications in $\mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}S_N^y)$.

Note that $\tilde{\iota}$ identifies the operators $\lambda_i, e_{jj} \otimes t, e_{kl}e_{lk} \in U_{\mathbf{0}}$ with the left multiplications by x_i, y_j, s_{kl}, respectively. Since x_i, y_j, s_{kl} generate the $\mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}S_N^y)$, the lemma is proved. □

In particular, by Lemma 3.3 the image of the Bethe subalgebra $\tilde{\iota}(\mathcal{B}_N)$ is a commutative subalgebra of $\mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}S_N^y)$.

Theorem 2.9 is equivalent to the following.

Corollary 3.4. *The subalgebras $\alpha(\mathcal{Z}_N)$ and $\tilde{\iota}(\mathcal{B}_N)$ of the algebra $\mathbb{C}[\boldsymbol{x}] \otimes (\mathbb{C}[\boldsymbol{y}] \ltimes \mathbb{C}S_N^y)$ coincide.* □

3.3. **The spherical subalgebra.** Recall that $e \in H_N$ is the symmetrizing element, see (2.2). The *spherical subalgebra* U_N is given by

$$U_N = eH_Ne \subset H_N.$$

We have the Satake homomorphism:

$$s: \ \mathcal{Z}_N \to H_N, \qquad c \mapsto c\, e.$$

Let K be the $N \times N$ matrix with all entries 1. Let $X = \mathrm{diag}(x_1, \dots, x_N)$, $Y = \mathrm{diag}(y_1, \dots, y_N)$ be diagonal $N \times N$ matrices.

Define the *universal spherical polynomial*

$$\mathcal{P}^U =: \mathrm{rdet}((v-X)(u-Y)-K) : e.$$

We have

$$s(\mathcal{P}^{\mathcal{Z}}) = \mathcal{P}^U.$$

Theorem 3.5. *The coefficients of the universal spherical polynomial $c_{ij}e$ generate the spherical subalgebra U_N. In particular, the Satake homomorphism is an isomorphism.*

Proof. The theorem follows from Theorem 2.5. □

The fact that Satake homomorphism is an isomorphism is not new, see [EG].

We use the isomorphism ι to identify the spherical subalgebra with a subspace of $\mathcal{V}_{\mathbf{1}}$. The left and right multiplications by $\sigma \in S_N$ considered as elements of H_N correspond to two actions of the symmetric group S_N on $\mathcal{V}$ which we call the *left and right actions*. These actions are defined as follows. The left action permutes the variables λ_i and vectors ϵ_j:

$$\sigma^L\bigl(p(\boldsymbol{z}, \lambda_1, \dots, \lambda_N)\, \epsilon_{i_1} \otimes \dots \otimes \epsilon_{i_N}\bigr) \;=\; p(\boldsymbol{z}, \lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)})\, \epsilon_{\sigma(i_1)} \otimes \dots \otimes \epsilon_{\sigma(i_N)}.$$

The left action restricted on $V^{\otimes N}$ coincides with the standard Weyl group action on representations of $\mathfrak{gl}_N$.

The right action permutes the variables z_i and the factors of V^N:

$$\sigma^R\bigl(p(z_1, \dots, z_N, \boldsymbol{\lambda})\, \epsilon_{i_1} \otimes \dots \otimes \epsilon_{i_N}\bigr) \;=\; p(z_{\sigma(1)}, \dots, z_{\sigma(N)}, \boldsymbol{\lambda})\, \epsilon_{i_{\sigma^{-1}(1)}} \otimes \dots \otimes \epsilon_{i_{\sigma^{-1}(N)}}.$$

Clearly the left and the right actions of S_N commute.

The space $\mathcal{V}_{\mathbf{1}}$ is invariant under the left and right S_N actions. Denote by $\mathcal{V}_{\mathbf{1}}^{S^L}$, $\mathcal{V}_{\mathbf{1}}^{S^R}$ the subspaces of invariants in $\mathcal{V}_{\mathbf{1}}$ with respect to the left and right actions respectively. Denote also by $\mathcal{V}_{\mathbf{1}}^{S^R \times S^L} = \mathcal{V}_{\mathbf{1}}^{S^L} \cap \mathcal{V}_{\mathbf{1}}^{S^R}$, the subspace of invariants with respect to both actions.

Lemma 3.6. *For any $v \in \mathcal{V}_{\mathbf{1}}$, $\sigma \in S_N$, we have*

$$\iota(\sigma^L v) = \sigma \iota v, \qquad \iota(\sigma^R v) = (\iota v)\sigma^{-1}.$$

Moreover

$$\iota(\mathcal{V}_{\mathbf{1}}^{S^R}) = He \subset H, \qquad \iota(\mathcal{V}_{\mathbf{1}}^{S^L}) = eH \subset H, \qquad \iota(\mathcal{V}_{\mathbf{1}}^{S^L \times S^R}) = U_N \subset H.$$

Proof. The lemma is straightforward. □

Corollary 3.7. *The Bethe algebra $\bar{\mathcal{B}}_N$ is isomorphic to the spherical subalgebra $U_N \subset H_N$. Moreover, the space $\mathcal{V}_{\mathbf{1}}^{S^L \times S^R}$ is a cyclic $\bar{\mathcal{B}}_N$-module which is identified with the regular representation of U_N by the isomorphism ι.* □

3.4. **Action of the Cherednik algebra.** We identify the space $\mathcal{V}_{\mathbf{1}}^{S^R} = \iota^{-1}(He)$ with the space $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]$ and compute the left action of the Cherednik algebra. Define the projection map:

$$\mathrm{pr} : \mathcal{V}_{\mathbf{1}} \to \mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}], \qquad \sum_{\sigma\in S_N} p_\sigma(\boldsymbol{z},\boldsymbol{\lambda})\epsilon_\sigma \mapsto p_{id}(\boldsymbol{z},\boldsymbol{\lambda}).$$

Lemma 3.8. *We have the isomorphisms of vector spaces:*

$$\begin{aligned} \mathrm{pr} &: \mathcal{V}_{\mathbf{1}}^{S^R} \to \mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}], \\ \mathrm{pr} &: \mathcal{V}_{\mathbf{1}}^{S^L\times S^R} \to P_N. \end{aligned}$$

Proof. The lemma is straightforward. □

In particular, we have

$$(\mathrm{pr}\,\iota^{-1}):\ H_N e \to \mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}], \qquad q(\boldsymbol{x})p(\boldsymbol{y})e \mapsto q(\boldsymbol{\lambda})p(\boldsymbol{z}).$$

Set

$$\mathcal{K}_i = z_i + \sum_{j,\ j\neq i} s_{ij}^z \,\frac{1}{\lambda_i-\lambda_j}(1-s_{ij}^\lambda). \tag{3.1}$$

Proposition 3.9. *We have*

$$(\mathrm{pr}\,\iota^{-1})x_i = \lambda_i(\mathrm{pr}\,\iota^{-1}), \qquad (\mathrm{pr}\,\iota^{-1})s_{ij} = s_{ij}^{z,\lambda}(\mathrm{pr}\,\iota^{-1}), \qquad (\mathrm{pr}\,\iota^{-1})y_i = \mathcal{K}_i(\mathrm{pr}\,\iota^{-1}).$$

In particular, the assignment $x_i = \lambda_i$, $y_i = \mathcal{K}_i$ and $s_{ij} = s_{ij}^{z,\lambda}$ defines a left action of H_N on $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]$ such that $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]$ is canonically identified with $H_N e$ as a left H_N-module.

Proof. The first and the second equalities are clear. To compute the action of y_1 on $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]$ it is sufficient to compute it on $\prod_{i=2}^N z_i^{a_i}$, since $\sum_{i=1}^N x_i$ and $\sum_{i=1}^N y_i$ are central elements. This calculation is straightforward. □

Remark 3.10. If $p(\boldsymbol{y})$ is a symmetric polynomial in $y_1,\dots,y_N$, then it is central, $p(\boldsymbol{y}) \in \mathcal{Z}_N$, and therefore it acts on $\mathbb{C}[\boldsymbol{z},\boldsymbol{\lambda}]$ as the operator of multiplication by $p(\boldsymbol{z})$.

4. Calogero-Moser spaces

4.1. **The definition.** Let

$$\tilde{C}_N = \{(Z,\Lambda) \in \mathfrak{gl}_N \times \mathfrak{gl}_N \mid \mathrm{rk}([Z,\Lambda]+1) = 1\}.$$

The group GL_N of complex invertible matrices acts on $\tilde{C}_N$ by simultaneous conjugation and the action is free and proper, see [Wi]. The quotient space C_N is called the *N-th Calogero-Moser space.* For $(Z,\Lambda)\in\tilde{C}_N$, we write $(Z,\Lambda)\in C_N$ meaning the orbit of GL_N containing (Z,Λ).

The Calogero-Moser space C_N is a smooth complex affine variety of dimension $2N$, see [Wi]. Let $\mathcal{O}_N^C$ be the *algebra of regular functions on* $\mathbb{C}^N$. It is defined as

follows. Let z_{ij}, λ_{ij} be the matrix entries of Z, Λ considered as functions on $\tilde{C}_N$. Let $\mathbb{C}[z_{ij}, \lambda_{ij}]^{GL_N}$ be the algebra of polynomials in x_{ij}, z_{ij} invariant with respect to the action of GL_N. Let $I \subset \mathbb{C}[z_{ij}, \lambda_{ij}]^{GL_N}$ be the ideal of the invariant polynomials which vanish on $\tilde{C}_N$. Then

$$\mathcal{O}_N^C = \mathbb{C}[z_{ij}, \lambda_{ij}]^{GL_N}/I.$$

4.2. **The Calogero-Moser Ψ-function.** Let $Z = (z_{ij})$, $\Lambda = (\lambda_{ij})$. Define the *Calogero-Moser Ψ-function* by

$$\Psi^C = \det(1 - (v - \Lambda)^{-1}(u - Z)^{-1}).$$

The Calogero-Moser Ψ-function (more precisely, the closely related to it stationary Baker function) was introduced in [Wi].

The Calogero-Moser Ψ-function is a formal power series in u^{-1} and v^{-1} with coefficients in $\mathcal{O}_N^C$. Write

$$\Psi^C = 1 + \sum_{i,j=1}^{\infty} \psi_{ij}^C u^{-i} v^{-j}, \qquad \psi_{ij}^C \in \mathcal{O}_N^C.$$

Lemma 4.1. *The algebra $\mathcal{O}_N^C$ of regular functions on C^N is generated by ψ_{ij}^C, $i, j \in \mathbb{Z}_{>0}$.*

Proof. For $\boldsymbol{i} = (i_1, \dots, i_k)$, $\boldsymbol{j} = (j_1, \dots, j_k)$, let $T_{\boldsymbol{i},\boldsymbol{j}} = \Lambda^{i_1} Z^{j_1} \dots \Lambda^{i_k} Z^{i_k}$. According to the standard theory of invariants, the algebra $\mathbb{C}[z_{ij}, \lambda_{ij}]^{GL_N}$ is generated by functions $\mathrm{tr}(T_{\boldsymbol{i},\boldsymbol{j}})$, see [W].

Define a $\mathbb{Z}_{\geqslant 0}$ filtration on algebra $\mathcal{O}_N^C$ by letting $|\boldsymbol{i}| = \sum_{s=1}^k i_s$, $|\boldsymbol{j}| = \sum_{s=1}^k j_s$ and $\deg \mathrm{tr}\, T_{\boldsymbol{i},\boldsymbol{j}} = |\boldsymbol{i}| + |\boldsymbol{j}|$. We say $\deg F \leqslant s$ if F can be written as a linear combination of products of $\mathrm{tr}\, T_{\boldsymbol{i},\boldsymbol{j}}$ with degree of each factor at most s.

We claim that

$$\mathrm{tr}\, T_{\boldsymbol{i},\boldsymbol{j}} = \mathrm{tr}\, T_{|\boldsymbol{i}|,|\boldsymbol{j}|} + ...,$$

where the dots denote the terms of degree less than $|\boldsymbol{i}| + |\boldsymbol{j}|$.

Given that claim, the proof of the lemma is similar to the proof of Lemma 2.3.

To prove the claim, let $K = 1 - [\Lambda, Z]$. Since the rank of K is one and $\mathrm{tr}\, K = N$, we see that

$$\mathrm{tr}(T_{\boldsymbol{i},\boldsymbol{j}} K T_{\boldsymbol{i}',\boldsymbol{j}'} K) = \frac{1}{N^2} \mathrm{tr}(K T_{\boldsymbol{i},\boldsymbol{j}} K K T_{\boldsymbol{i}',\boldsymbol{j}'} K) =$$

$$\frac{1}{N^2} \mathrm{tr}(K T_{\boldsymbol{i},\boldsymbol{j}} K) \,\mathrm{tr}(K T_{\boldsymbol{i}',\boldsymbol{j}'} K) = \mathrm{tr}(T_{\boldsymbol{i},\boldsymbol{j}} K) \,\mathrm{tr}(T_{\boldsymbol{i}',\boldsymbol{j}'} K).$$

It follows that

$$\mathrm{tr}(T_{\boldsymbol{i},\boldsymbol{j}}[\Lambda, Z]) = \mathrm{tr}(\Lambda^{|\boldsymbol{i}|} Z^{|\boldsymbol{j}|}[\Lambda, Z]) + \dots,$$

where the dots denote the terms of degree less than $|\boldsymbol{i}| + |\boldsymbol{j}| + 2$.

Therefore it is sufficient to prove that for $i, j \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathrm{tr}(\Lambda^i Z^j \Lambda Z) = \mathrm{tr}(\Lambda^{i+1} Z^{j+1}) + \dots,$$

where the dots denote the terms of degree less than $i+j+2$.

By the cyclic property of the trace we have

$$\operatorname{tr}(\Lambda^i Z^j \Lambda Z) - \operatorname{tr}(\Lambda^i \Lambda Z^j Z) = -\operatorname{tr}(\Lambda^i Z^j K) + \dots,$$

where the dots denote the terms of degree less than $i+j+2$. On the other hand, commuting Λ through Z^j, we obtain

$$\operatorname{tr}(\Lambda^i Z^j \Lambda Z) - \operatorname{tr}(\Lambda^i \Lambda Z^j Z) = \sum_{k=0}^{j-1} \operatorname{tr}(\Lambda^i Z^k K Z^{j-k}) + \dots = j \operatorname{tr}(\Lambda^i Z^j K) + \dots,$$

where the dots denote the terms of degree less than $i+j+2$. The claim follows. □

Define the *Calogero-Moser universal polynomial* by

$$\mathcal{P}^C = \det((v-\Lambda)(u-Z)-1).$$

Write

$$\Psi^C = \sum_{i,j=0}^{N} m_{ij}\; u^{N-i} v^{N-j}, \qquad m_{ij} \in \mathcal{O}_N^C.$$

Lemma 4.2. *The algebra $\mathcal{O}_N^C$ of regular functions on C^N is generated by m_{ij}, $i,j = 0,1,\dots,N$.*

Proof. We have

$$\det(u-Z) = u^N + \sum_{i=1}^{N} m_{i0} u^{N-i}, \qquad \det(v-\Lambda) = v^N + \sum_{j=1}^{N} m_{0j} v^{N-j}.$$

In particular, the coefficients of $\det(u-Z)^{-1}$ and $\det(v-\Lambda)^{-1}$ are in the algebra generated by m_{ij}. Since

$$\det(u-Z)^{-1} \det(v-\Lambda)^{-1}\; \mathcal{P}^C = \Psi^C,$$

the lemma follows from Lemma 4.1. □

4.3. **The Bethe algebra and the algebra of functions on the Calogero-Moser space.** Recall that $\bar{\mathcal{B}}_N$ is the image of the Bethe algebra $\mathcal{B}_N \in U(\mathfrak{gl}_N[t])[\boldsymbol{\lambda}]$ in $\mathrm{End}(\mathcal{V}_{\mathbf{1}})$ and $\bar{b}_{ij}$ are the generators of $\bar{\mathcal{B}}_N$.

Define the map

$$\tau_{OB}:\ \mathcal{O}_N^C \to \bar{\mathcal{B}}_N, \qquad m_{ij} \mapsto \bar{b}_{ij}.$$

Theorem 4.3. *The map τ is a well-defined algebra isomorphism.*

Theorem 4.3 is proved in Section 5.

Corollary 4.4. *The space $\mathcal{V}_{\mathbf{1}}^{S^L \times S^R}$ is a cyclic $\bar{\mathcal{B}}_N$-module which is isomorphic to the regular representation of $\mathcal{O}_N^C$.* □

5. Proof of Theorem 4.3

5.1. **Spaces of quasi-exponentials.** For any $n \in \mathbb{Z}_{>0}$ and complex numbers $\lambda_1^0, \dots, \lambda_n^0$, we call a complex vector space W of dimension n with a basis of the form $q_1(u)e^{\lambda_1^0 u}$, $\dots, q_n(u)e^{\lambda_n^0 u}$, where $q_i(u) \in \mathbb{C}[u]$, $i = 1, \dots, n$, a *space of quasi-exponentials with exponents* $\boldsymbol{\lambda}^0 = (\lambda_1^0, \dots, \lambda_n^0)$.

Let

$$\mathrm{Wr}_W \,=\, c\,\det\Big(\partial^{j-1}\big(q_i(u)e^{\lambda_i^0 u}\big)\Big)_{i,j=1}^{n} e^{-\sum_{i=1}^n \lambda_i^0 u}$$

where $c \in \mathbb{C}$ is a non-zero constant such that Wr_W is a monic polynomial. The polynomial Wr_W does not depend on the choice of a basis in W and is called the *Wronskian of* W. We denote the $\deg \mathrm{Wr}_W$ simply by $\deg W$.

Zeros of Wronskian Wr_W are called *singular points* of W. The number of singular points counted with multiplicity equals $\deg W$.

We denote by $\mathcal{D}_W^{\mathcal{W}}$ the monic scalar differential operator of order n with kernel W. The operator $\mathcal{D}_W^{\mathcal{W}}$ is Fuchsian with singular points exactly at the singular points of W and infinity.

Write

$$\mathcal{D}_W^{\mathcal{W}} = \partial^n + \sum_{i=1}^n G_{i,W}(u)\partial^{n-i}.$$

We have

$$v^n + \sum_{i=1}^n G_{i,W}(u)v^{n-i} = \frac{c}{\mathrm{Wr}_W}\,\cdot\,\det\begin{pmatrix} q_1(u)e^{\lambda_1^0 u} & \partial(q_1(u)e^{\lambda_1^0 u}) & \dots & \partial^n(q_1(u)e^{\lambda_1^0 u}) \\ \dots & \dots & \dots & \dots \\ q_n(u)e^{\lambda_n^0 u} & \partial(q_n(u)e^{\lambda_n^0 u}) & \dots & \partial^n(q_n(u)e^{\lambda_n^0 u}) \\ 1 & v & \dots & v^n \end{pmatrix}.$$

We call the function

$$\Psi_W^{\mathcal{W}} \,=\, \Big(v^n + \sum_{i=1}^n G_{i,W}(u)v^{n-i}\Big)\,\prod_{i=1}^n (v-\lambda_i^0)^{-1}$$

the *Ψ-function of the space W*. The Ψ-function is a formal power series in u^{-1} and v^{-1} with complex coefficients. Moreover, it has the form

$$\Psi_W^{\mathcal{W}} = 1 + \sum_{i,j=1}^{\infty} \psi_{ij,W}^{\mathcal{W}} u^{-i}v^{-j}, \qquad \psi_{ij,W}^{\mathcal{W}} \in \mathbb{C}.$$

Let W_1 and W_2 be spaces of quasi-exponentials of possibly different dimensions. We call the spaces W_1 and W_2 *equivalent* if $\Psi_{W_1}^{\mathcal{W}} = \Psi_{W_2}^{\mathcal{W}}$. This defines an equivalence relation on the set of spaces of quasi-exponentials.

We call a space of quasi-exponentials W *minimal* if W does not contain a function of the form $e^{\lambda^0 u}$ with $\lambda^0 \in \mathbb{C}$.

For $\lambda^0 \in \mathbb{C}$, let $W(\lambda^0) \subset W$ be the subspace spanned by all function in W of the form $q(u)e^{\lambda^0 u}$, $q(u) \in \mathbb{C}[u]$. Then $W(\lambda^0)$ is also a space of quasi-exponentials. We call a space of quasi-exponentials W *canonical* if for every $\lambda^0 \in \mathbb{C}$ we have the equality

$$\dim W(\lambda^0) = \deg W(\lambda^0).$$

Note that for canonical spaces of quasi-exponentials we have $\dim W = \deg W$ and for minimal spaces we have $\dim W \leqslant \deg W$.

Lemma 5.1. *Each equivalence class of spaces of quasi-exponentials contains exactly one minimal and exactly one canonical space of quasi-exponentials. If W_1 and W_2 are equivalent spaces of quasi-exponentials, then $\mathrm{Wr}_{W_1} = \mathrm{Wr}_{W_2}$ in particular, $\deg W_1 = \deg W_2$. If in addition, W_1 is minimal and $W_2 \neq W_1$, then $\dim W_1 < \dim W_2$.*

Proof. The lemma is straightforward. □

Define the *degree of an equivalence class of spaces of quasi-exponentials* as the degree of any of representative of this class. Denote the set of all equivalence classes of quasi-exponentials of degree N by Q_N.

We call a space of quasi-exponentials *generic* if all λ_i^0 are distinct and all $q_i(u)$ are linear polynomials. A space of quasi-exponentials is generic if and only if it is both canonical and minimal.

To a point $(\boldsymbol{h}^0, \boldsymbol{\lambda}^0) \in \mathbb{C}^N \times \mathbb{C}^N$ such that all λ_i^0 are distinct, we associate a generic space of quasi-exponentials

$$W_{\boldsymbol{h}^0,\boldsymbol{\lambda}^0} = \langle (u - h_i^0)\ e^{\lambda_i^0 u},\ i = 1, \dots, N \rangle.$$

Generic spaces of quasi-exponentials of degree N are in a bijective correspondence with $S_N^{h,\lambda}$ orbits of points $(\boldsymbol{h}^0, \boldsymbol{\lambda}^0) \in \mathbb{C}^N \times \mathbb{C}^N$ such that all λ_i^0 are distinct.

5.2. **The Bethe ansatz.** Let $\boldsymbol{z}^0 = (z_1^0, \dots, z_N^0)$ and $\boldsymbol{\lambda}^0 = (\lambda_1^0, \dots, \lambda_N^0)$ be sequences of complex numbers.

Consider $V^{\otimes N}$ as the tensor product of evaluation $\mathfrak{gl}_N[t]$-modules with evaluation parameters $z_1^0, \dots, z_N^0$. Namely, the action of $g \otimes t^k \in \mathfrak{gl}_N[t]$ is given by

$$(g \otimes t^k)(v_1 \otimes \cdots \otimes v_N) = \sum_{i=1}^{N} (z_i^0)^k\ v_1 \otimes \cdots \otimes g v_i \otimes \cdots \otimes v_N.$$

Let λ_i act on $V^{\otimes N}$ as multiplication by λ_i^0:

$$\lambda_i (v_1 \otimes \cdots \otimes v_N) = \lambda_i^0\ v_1 \otimes \cdots \otimes v_N.$$

Then $(V^{\otimes N})_{\boldsymbol{1}}$ is a $\mathcal{B}_N$-module which we denote by $(V^{\otimes N}(\boldsymbol{z}^0, \boldsymbol{\lambda}^0))_{\boldsymbol{1}}$.

Let $v \in (V^{\otimes N}(\boldsymbol{z}^0, \boldsymbol{\lambda}^0))_{\boldsymbol{1}}$ be an eigenvector of the Bethe algebra, $B_{ij} v = B_{ij,v} v$, where $B_{ij,v} \in \mathbb{C}$. Consider the scalar differential operator

$$\mathcal{D}_v^{\mathcal{B}} = \partial^N + \sum_{i=1}^{N} \sum_{j=0}^{\infty} B_{ij,v} u^{-j} \partial^{N-i}.$$

Lemma 5.2 ([MTV7]). *If $z_i^0 \neq z_j^0$ and $\lambda_i^0 \neq \lambda_j^0$ whenever $i \neq j$, then the kernel of the operator $\mathcal{D}_v^{\mathcal{B}}$ is a generic space of quasi-exponentials of degree N with exponents $\boldsymbol{\lambda}^0$ and singular points $\boldsymbol{z}^0$.* $\square$

We need a statement which says that generically the $\mathcal{B}_N$-module $(V^{\otimes N}(\boldsymbol{z}^0, \boldsymbol{\lambda}^0))_{\mathbf{1}}$ is the sum of non-isomorphic one-dimensional modules which are in a bijective correspondence with generic spaces of quasi-exponentials of degree N with exponents $\boldsymbol{\lambda}^0$ and singular points $\boldsymbol{z}^0$. Such a statement is proved by the Bethe ansatz method.

Recall that the generic spaces of quasi-exponentials $W_{\boldsymbol{h}^0,\boldsymbol{\lambda}^0}$ of degree N are parameterized by $(\boldsymbol{h}^0, \boldsymbol{\lambda}^0) \in \mathbb{C}^N \times \mathbb{C}^N$ such that λ_i^0 are all distinct, see Section 5.1.

Lemma 5.3. *There exist Zariski open S_N invariant subsets Θ and Ξ of $\mathbb{C}^N \times \mathbb{C}^N$ such that*

(i) *For any $(\boldsymbol{z}^0, \boldsymbol{\lambda}^0) \in \Theta$, we have $z_i^0 \neq z_j^0$ and $\lambda_i^0 \neq \lambda_j^0$ for $i \neq j$, and there exists a basis for $(V^{\otimes N}(\boldsymbol{z}^0, \boldsymbol{\lambda}^0))_{\mathbf{1}}$ such that every basis vector v is an eigenvector of the Bethe algebra, and $\mathcal{D}_v^{\mathcal{B}} = \mathcal{D}^{\mathcal{W}}_{W_{\boldsymbol{h}^0,\boldsymbol{\lambda}^0}}$, where $(\boldsymbol{h}^0, \boldsymbol{\lambda}^0) \in \Xi$.*

(ii) *For any $(\boldsymbol{h}^0, \boldsymbol{\lambda}^0) \in \Xi$, there exists a unique up to a permutation $(\boldsymbol{z}^0, \boldsymbol{\lambda}^0) \in \Theta$ and a unique up to proportionality vector $v \in (V^{\otimes N}(\boldsymbol{z}^0, \boldsymbol{\lambda}^0))_{\mathbf{1}}$ such that v is an eigenvector of the Bethe algebra and $\mathcal{D}_v^{\mathcal{B}} = \mathcal{D}^{\mathcal{W}}_{W_{\boldsymbol{h}^0,\boldsymbol{\lambda}^0}}$.*

Proof. This lemma is a special case of Lemma 6.1 in [MTV3]. $\square$

5.3. **The modules $\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0)$.** Let $\sigma_i(\boldsymbol{z})$ and $\sigma_i(\boldsymbol{\lambda})$, $i = 1, \dots, N$, be the elementary symmetric functions

$$\prod_{i=1}^{N}(u - z_i) = u^N + \sum_{i=1}^{N}(-1)^i \sigma_i(\boldsymbol{z}) u^{N-i}, \qquad \prod_{i=1}^{N}(v - \lambda_i) = v^N + \sum_{i=1}^{N}(-1)^i \sigma_i(\boldsymbol{\lambda}) v^{N-i}.$$

For $(\boldsymbol{a}^0, \boldsymbol{b}^0) \in \mathbb{C}^N \times \mathbb{C}^N$ let $I_{\boldsymbol{a}^0,\boldsymbol{b}^0} \subset \mathbb{C}[\boldsymbol{z}, \boldsymbol{\lambda}]$ be the ideal generated by the functions $\sigma_i(\boldsymbol{z}) = a_i^0$ and $\sigma_i(\boldsymbol{\lambda}) = b_i^0$, $i = 1, \dots, N$.

Set
$$\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0) = \mathcal{V}_{\mathbf{1}}^{S^L \times S^R} / (\mathcal{V}_{\mathbf{1}}^{S^L \times S^R} \bigcap (V^{\otimes N})_{\mathbf{1}} \otimes I_{\boldsymbol{a}^0,\boldsymbol{b}^0}).$$
The action of the Bethe algebra $\mathcal{B}_N$ in $\mathcal{V}_{\mathbf{1}}$ induces an action of the algebra $\mathcal{B}_N$ in the space $\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0)$.

Let $(\boldsymbol{z}^0, \boldsymbol{\lambda}^0) \in \mathbb{C}^N \otimes \mathbb{C}^N$ be such that $\sigma_i(\boldsymbol{z}^0) = a_i^0$, $\sigma_i(\boldsymbol{\lambda}^0) = b_i^0$:

$$\prod_{i=1}^{N}(u - z_i^0) = u^N + \sum_{i=1}^{N}(-1)^i a_i^0 u^{N-i}, \qquad \prod_{i=1}^{N}(v - \lambda_i^0) = v^N + \sum_{i=1}^{N}(-1)^i b_i^0 v^{N-i}.$$

The following lemma is proved by standard methods.

Lemma 5.4. *We have* $\dim \mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0) = N!$. *If all z_i^0 are distinct and all λ_i^0 are distinct then the $\mathcal{B}_N$-modules $\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0)$ and $(V^{\otimes N}(\boldsymbol{z}^0, \boldsymbol{\lambda}^0))_{\mathbf{1}}$ are isomorphic.* $\square$

Remark 5.5. The $\mathcal{B}_N$-modules $\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0)$ and $(V^{\otimes N}(\boldsymbol{z}^0, \boldsymbol{\lambda}^0))_{\mathbf{1}}$ are not always isomorphic. For example, the former is always cyclic and the latter is not. If all λ_i^0 are distinct then the module $\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0)$ is isomorphic to a subspace in the Weyl module, see [MTV3]. It is interesting to understand the module $\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0)$ and it's precise relation to the module $(V^{\otimes N}(\boldsymbol{z}^0, \boldsymbol{\lambda}^0))_{\mathbf{1}}$ for all values of parameters.

5.4. **A relation of Calogero-Moser spaces to the spaces of quasi-exponentials.** Let $(Z, \Lambda) \in C_N$. Let the values of ψ_{ij}^C on (Z, Λ) be $\psi_{ij,Z,\Lambda}^C \in \mathbb{C}$. Then we obtain a formal power series with complex coefficients:

$$\Psi_{Z,\Lambda}^C = 1 + \sum_{i,j=1}^{\infty} \psi_{ij,Z,\Lambda}^C u^{-i} v^{-j}.$$

Theorem 5.6 ([Wi]). *For any $(Z, \Lambda) \in C_N$, there exists a space of quasi-exponentials W of degree N such that the exponents of W are eigenvalues of Z, the singular points of W are eigenvalues of Λ and $\Psi_{Z,\Lambda}^C = \Psi_W^{\mathcal{W}}$. Moreover, this establishes a bijective correspondence between points of C_N and the set Q_N of equivalence classes of spaces of quasi-exponentials of degree N.* □

We call $(Z, \Lambda) \in C_N$ a *generic point* if Z has a simple spectrum. The set of generic points is dense in C_N, see [Wi]. The generic points correspond to equivalence classes of quasi-exponentials of degree N which contain a generic space of quasi-exponentials.

5.5. **Proof of Theorem 4.3.** The proof is similar to the proof of Theorem 5.3 in [MTV2].

First we show that the map τ_{OB} is well defined. Let a polynomial $R(m_{ij})$ in generators m_{ij} be equal to zero in $\mathcal{O}_N^C$. We need to prove that $R(\bar{b}_{ij})$ is equal to zero in the algebra $\bar{\mathcal{B}}_N$. Consider $R(\bar{b}_{ij})$ as a polynomial in $z_1, \dots, z_N$ and $\lambda_1, \dots, \lambda_N$ with values in $\mathrm{End}\big((V^{\otimes N})_{\mathbf{1}}\big)$. Let Θ be as in Lemma 5.3, and $(\boldsymbol{z}^0, \boldsymbol{\lambda}^0) \in \Theta$. Then by part (i) of Lemma 5.3, the value of the polynomial $R(\bar{b}_{ij})$ at $z_1 = z_1^0, \dots, z_N = z_N^0$ and $\lambda_1 = \lambda_1^0, \dots, \lambda_N = \lambda_N^0$ equals zero. Hence, the polynomial $R(\bar{b}_{ij})$ equals zero identically.

Next we show that the map τ_{OB} is injective. Let a polynomial $R(m_{ij})$ in generators m_{ij} be a nonzero element of $\mathcal{O}_N^C$. Then the value of $R(m_{ij})$ at a generic point $(Z, \Lambda) \in C_N$ is not equal to zero. Then by part (ii) of Lemma 5.3, the polynomial $R(\bar{b}_{ij})$ is not identically equal to zero.

Finally, the map τ_{OB} is surjective since the elements $\bar{b}_{ij}$ generate the algebra $\bar{\mathcal{B}}_N$. □

Remark 5.7. Let $L_{\boldsymbol{\nu}^i}$, $i = 1, \dots, k$, be irreducible finite-dimensional $\mathfrak{gl}_N$-modules corresponding to partitions $\boldsymbol{\nu}^i$. The Bethe algebra $\mathcal{B}_N$ acts on the space $(\otimes_{i=1}^k L_{\boldsymbol{\nu}^i}) \otimes \mathbb{C}[z_1, \dots, z_k, \lambda_1, \dots, \lambda_N]$. Let $\tilde{B}_{ij}$ be the linear operators corresponding to the operators B_{ij} and let $\tilde{\mathcal{B}}_N$ be the algebra generated by $\tilde{B}_{ij}$.

Let

$$\Psi^{\tilde{\mathcal{B}}} = \Big(v^N + \sum_{i=1}^{N}\sum_{j=0}^{\infty} \tilde{B}_{ij} u^{-j} v^{N-i}\Big) \prod_{i=1}^{N} (v-\lambda_i)^{-1}.$$

Set $n = \sum_{i=1}^{k} |\boldsymbol{\nu}^i|$. Then we have a map $\mathcal{O}_n^C \to \tilde{\mathcal{B}}_N$, which sends the coefficients of Ψ^C to the corresponding coefficients of $\Psi^{\tilde{\mathcal{B}}}$. Similarly to Theorem 4.3, using the results of [MTV2], one can show that this map is a well defined homomorphism of algebras. However, it is neither injective nor surjective in general, see Section 5.2 in [MTV6].

6. Corollaries of Theorems 2.9, 4.3

6.1. **Regular functions on the Calogero-Moser space and the center of the Cherednik algebra.** Define an algebra homomorphism

$$\tau_{OZ} : \mathcal{O}_N^C \to \mathcal{Z}_N, \qquad m_{ij} \mapsto c_{ij}.$$

Corollary 6.1. *The map τ_{CZ} is a well-defined algebra isomorphism and $\tau_{OZ} = \tau_{BZ} \circ \tau_{OB}$.*

Proof. By Theorems 4.3 and 2.9, the maps τ_{BZ} and τ_{OB} are algebra isomorphisms such that $\tau_{OB}(m_{ij}) = \bar{b}_{ij}$ and $\tau_{BZ}(\bar{b}_{ij}) = c_{ij}$. The claim follows. □

The fact that algebras $\mathcal{O}_N^C$ and $\mathcal{Z}_N$ are isomorphic is proved by a different method in [EG].

6.2. **Bijections.** Recall that we have the following sets.

- The Calogero-Moser space C_N.
- The set Q_N of equivalence classes of spaces of quasi-exponentials of degree N.
- The set R_N of isomorphisms classes of the irreducible representations of the Cherednik algebra U_N.

There are well known bijections between these three sets. The bijection between C_N and Q_N is contained in [Wi], see also Theorem 5.6. The bijection between C_N and R_N is described in [EG]. We add one more set to this list:

- The set of eigenvectors of the Bethe algebra $\mathcal{B}_N$ up to a multiplication by a non-zero number in

$$\boldsymbol{\mathcal{V}}_{\mathbf{1}} = \bigoplus_{(\boldsymbol{a}^0,\boldsymbol{b}^0)\in\mathbb{C}^N\times\mathbb{C}^N} \mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0,\boldsymbol{b}^0).$$

We denote this set by E_N.

We describe the bijections of E_N to the first three sets. Let $v \in \mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0,\boldsymbol{b}^0) \subset \boldsymbol{\mathcal{V}}_{\mathbf{1}}$ be an eigenvector of the Bethe algebra $\mathcal{B}_N$. Let $B_{ij,v} \in \mathbb{C}$ be the corresponding eigenvalues: $B_{ij}v = B_{ij,v}v$.

Note that the action of the algebra $\mathcal{B}_N$ in $\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0,\boldsymbol{b}^0)$ factors through the action of the algebra $\bar{\mathcal{B}}_N$. In particular, by Theorem 4.3, the algebra $\mathcal{O}_N^C$ acts on $\mathcal{V}_{\mathbf{1}}(\boldsymbol{a}^0,\boldsymbol{b}^0)$.

Moreover, an eigenvector v of the Bethe algebra $\mathcal{B}_N$ defines an algebra homomorphism $\chi_v : \mathcal{O}_N^C \to \mathbb{C}$.

Corollary 6.2. *If $v, w \in \boldsymbol{\mathcal{V}}_{\mathbf{1}}$ are eigenvectors of the Bethe algebra $\mathcal{B}_N$ and $B_{ij,w} = B_{ij,v}$ for all i, j then $w = cv$ for some $c \in \mathbb{C}$.*

Proof. By Corollary 4.4, the space $V_{\mathbf{1}}^{S^R \times S^L}$ is a regular representation of the algebra $\mathcal{O}_C^N$. The regular and coregular representations of the algebra $\mathcal{O}_C^N$ are isomorphic. Therefore the kernel of the ideal $\operatorname{Ker}\chi_v$ is one-dimensional. □

Let $\nu_C : E_N \to C_N$ be the map which sends v to the point in C_N corresponding to the maximal ideal $\operatorname{Ker}\chi_v \subset \mathcal{O}_N^C$.

Corollary 6.3. *The map ν_C is a bijection.*

Proof. The corollary follows from Theorem 4.3 and Corollary 4.4. □

Let $\nu_Q : E_N \to Q_N$ be the map which sends v to the kernel W_v of the differential operator

$$\mathcal{D}_v^{\mathcal{B}} = \partial^N + \sum_{i=1}^{N} \sum_{j=0}^{\infty} B_{ij,v} u^{-j} v^{N-i}.$$

Corollary 6.4. *For every eigenvector $v \in \boldsymbol{\mathcal{V}}_{\mathbf{1}}$ of the algebra $\mathcal{B}_N$, the W_v is a canonical space of quasi-exponential of degree N. The map ν_Q is a bijection.*

Proof. The space W_v is a space of quasi-exponentials of degree N by Corollary 6.3 and Theorem 5.6. This space is generic for generic values of $\boldsymbol{a}^0, \boldsymbol{b}^0$, see Lemma 5.2. It follows by continuity that W_v is a canonical space of quasi-exponentials of degree N. Therefore the map ν_Q is well defined. The map ν_Q a bijection by Corollary 6.3 and Theorem 5.6. □

Let

$$\tilde{\mathcal{V}}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0) = \mathcal{V}_{\mathbf{1}}^{S^R} / (\mathcal{V}_{\mathbf{1}}^{S^R} \bigcap (V^{\otimes N})_{\mathbf{1}} \otimes I_{\boldsymbol{a}^0, \boldsymbol{b}^0}).$$

and

$$\tilde{\boldsymbol{\mathcal{V}}}_{\mathbf{1}} = \bigoplus_{(\boldsymbol{a}^0, \boldsymbol{b}^0) \in \mathbb{C}^N \times \mathbb{C}^N} \tilde{\mathcal{V}}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0).$$

Clearly, we have an inclusion $\boldsymbol{\mathcal{V}}_{\mathbf{1}} \subset \tilde{\boldsymbol{\mathcal{V}}}_{\mathbf{1}}$.

The space $\tilde{\boldsymbol{\mathcal{V}}}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0)$ is the left H_N-module. In particular, an eigenvector of the Bethe algebra $v \in \boldsymbol{\mathcal{V}}_{\mathbf{1}} \subset \tilde{\boldsymbol{\mathcal{V}}}_{\mathbf{1}}$ defines an algebra homomorphism $\chi_v : \mathcal{Z}_N \to \mathbb{C}$.

Let $\nu_R : E_N \to R_N$ be the map which sends v to the H_N-submodule M_v of $\tilde{\boldsymbol{\mathcal{V}}}_{\mathbf{1}}(\boldsymbol{a}^0, \boldsymbol{b}^0)$ generated by v.

Corollary 6.5. *For every eigenvector $v \in \boldsymbol{\mathcal{V}}_{\mathbf{1}}$ of algebra $\mathcal{B}_N$, M_v is an irreducible representation corresponding to the central character $\chi_v : \mathcal{Z}_N \to \mathbb{C}$. The map ν_R is a bijection.*

Proof. By Theorem 2.6 the irreducible representations of H_N are determined by the central characters $\chi : \mathcal{Z}_N \to \mathbb{C}$ and have the form $H_N e \otimes_{\mathcal{Z}_N} \chi$. Recall that $\mathcal{V}^{S^R}_{\mathbf{1}}$ is identified with $H_N e$, see Lemma 3.6. By Corollary 6.1, the central characters of H_N are in a bijective correspondence with the points of the Calogero-Moser space C_N and by Corollary 6.3 the points of the Calogero-Moser space are in a bijective correspondence with the set E_N. The corollary follows. $\square$

6.3. **Example $N = 2$.** The algebra $\bar{B}_2 \simeq \mathcal{O}_2 \simeq \mathcal{Z}_2$ can be described by generators and relations as follows

$$\mathbb{C}[g_1, g_2, h_1, h_2, T]/(T^2 - h_1 g_1 T + (g_1^2 - 2g_2)h_2 + (h_1^2 - 2h_2)g_2 - 1).$$

It is a free module of rank 2 over the subalgebra $\mathbb{C}[g_1, g_2, h_1, h_2]$ generated by 1 and T. We describe the corresponding universal polynomials and generators for all three algebras.

The universal central polynomial has the form

$$\begin{aligned}
&\mathcal{P}^{\mathcal{Z}} = \\
&(1 - (v - x_1)(u - y_1) - (v - x_2)(u - y_2) + (v - x_1)(v - x_2)(u - y_1)(u - y_2)) - s_{12} = \\
&v^2u^2 - (y_1 + y_2)v^2u - (x_1 + x_2)vu^2 + y_1y_2v^2 + x_1x_2u^2 + ((x_1 + x_2)(y_1 + y_2) - 2)vu - \\
&\qquad ((x_1 + x_2)y_1y_2 - (y_1 + y_2))v - ((x_1x_2(y_1 + y_2) - (x_1 + x_2))u + \\
&\qquad 1 + x_1x_2y_1y_2 - x_1y_1 - x_2y_2 - s_{12}.
\end{aligned}$$

In particular the generators g_1, g_2, h_1, h_2, T of the center $\mathcal{Z}_2$ of H_2 are given by

$$x_1 + x_2, \quad y_1 + y_2, \quad x_1x_2, \quad y_1y_2, \quad x_1y_1 + x_2y_2 - s_{12}.$$

The Calogero-Moser universal polynomial has the form

$$\begin{aligned}
&\mathcal{P}^C = \det((v - \Lambda)(u - Z) - 1) = \\
&\qquad u^2v^2 - \mathrm{tr}(Z)v^2u - \mathrm{tr}(\Lambda)vu^2 + \det(Z)v^2 + \det(\Lambda)u^2 + (\mathrm{tr}(\Lambda)\,\mathrm{tr}(Z)vu + \\
&\qquad (\det(\Lambda)\,\mathrm{tr}(Z) - \mathrm{tr}(\Lambda))v + (\det(Z)\,\mathrm{tr}(\Lambda) - \mathrm{tr}(Z))u + 1 + \det(\Lambda Z) - \mathrm{tr}(\Lambda Z).
\end{aligned}$$

The generators g_1, g_2, h_1, h_2, T of the algebra $\mathcal{O}_2$ of the regular functions on C_2 are given by

$$\mathrm{tr}(\Lambda), \quad \mathrm{tr}(Z), \quad \det(\Lambda), \quad \det(Z), \quad \mathrm{tr}(\Lambda Z).$$

Note that $\det(\Lambda) = ((\mathrm{tr}(\Lambda))^2 - \mathrm{tr}(\Lambda^2))/2$, $\det(Z) = ((\mathrm{tr}(Z))^2 - \mathrm{tr}(Z^2))/2$.

The universal differential operator of B_2 has the form:

$$\begin{aligned}
\mathcal{D}^{\mathcal{B}} = \partial^2 - (\lambda_1 + \lambda_2 + e_{11}(u) + e_{22}(u))\partial + \\
(\lambda_1 + e_{11}(u))(\lambda_2 + e_{22}(u)) - e_{21}(u)e_{12}(u) - (e_{22}(u))'.
\end{aligned}$$

The universal Bethe polynomial has the form

$$\mathcal{P}^{\bar{\mathcal{B}}} = (u-z_1)(u-z_2)v^2 - ((\lambda_1+\lambda_2)(u-z_1)(u-z_2) + 2u - z_1 - z_2)v + 1 + \lambda_1\lambda_2 z_1 z_2 - \begin{pmatrix} \lambda_1 z_1 + \lambda_2 z_2 & -1 \\ -1 & \lambda_1 z_2 + \lambda_2 z_1 \end{pmatrix}.$$

Here we used the basis $\{\epsilon_{id} = \epsilon_1 \otimes \epsilon_2, \epsilon_{s_{12}} = \epsilon_2 \otimes \epsilon_1\}$. The space $\mathcal{V}_{\mathbf{1}}$ is a free $\mathbb{C}[\boldsymbol{z}, \boldsymbol{\lambda}]$-module of rank 2 with generators $\{\epsilon_{id}, \epsilon_{s_{12}}\}$, the action of $\bar{\mathcal{B}}_2$ commutes with multiplication by elements of $\mathbb{C}[\boldsymbol{z}, \boldsymbol{\lambda}]$.

The generators g_1, g_2, h_1, h_2, T of the image $\bar{\mathcal{B}}_2$ of the Bethe algebra $\mathcal{B}_2$ are given by

$$\lambda_1+\lambda_2, \quad z_1+z_2, \quad \lambda_1\lambda_2, \quad z_1z_2, \quad \begin{pmatrix} \lambda_1 z_1 + \lambda_2 z_2 & -1 \\ -1 & \lambda_1 z_2 + \lambda_2 z_1 \end{pmatrix}.$$

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